12 Infinite Sequences and Series

12.1 Sequences

A sequence is called monotonic if it is either increasing or decreasing, not both.
A sequence is bounded above if there exists M such that $a_n \leq M$ for all $n \geq 1$.
A sequence is bounded below if there exists m such that $a_n \geq m$ for all $n \geq 1$.
Every bounded, monotonic sequence is convergent.

12.2 Series

Geometric Series: $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$
Convergent if $|r| < 1$, then sum is $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$
Divergent if $|r| \geq 1$.
Telescoping Series: Write out several terms and look for a pattern of cancellation.
Test for Divergence: If $\lim_{n->\infty} a_n$ does not exist or $\lim_{n->\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.
WARNING: The Test for Divergence cannot prove that a series is convergent.

12.3 The Integral Tests and Estimates of Sums

Integral Test:
Preconditions: $f(x)$ is continuous, positive, and decreasing on $[1, \infty)$.
If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.
P-Series Test: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, diverges if $p \leq 1$
Remainder Estimate for the Integral Test: $\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$

12.4 The Comparison Tests

Comparison Test:
Preconditions: $\sum a_n$ and $\sum b_n$ are series with positive terms.
If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ converges.
If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ diverges.
Limit Comparison Test:
Preconditions: $\sum a_n$ and $\sum b_n$ are series with positive terms.
If $\lim_{n->\infty} \frac{a_n}{b_n} = C$ where C is finite and $C > 0$, then both series either converge or both diverge.

12.5 Alternating Series

$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b - b_2 + b_3 - b_4 + \dots$
If $b_{n+1} \leq b_n$ for all n and $\lim_{n->\infty} b_n = 0$ then the series converges.

12.6 Absolute Convergence and the Ratio and Root Tests

A series is absolutely convergent if $\sum |a_n|$ is convergent.
A series is conditionally convergent if it is convergent but not absolutely convergent.
If a series is absolutely convergent, then it is convergent.

Ratio Test:
If $\lim_{n->\infty} |\frac{a_{n+1}}{a_n}| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
If $\lim_{n->\infty} |\frac{a_{n+1}}{a_n}| = L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
If $\lim_{n->\infty} |\frac{a_{n+1}}{a_n}| = 1$, then the Ratio Test is inconclusive.
Root Test:
If $\lim_{n->\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
If $\lim_{n->\infty} \sqrt[n]{|a_n|} = L > 1$ or $lim_{n->\infty} \sqrt[n]{|a_n|}$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
If $\lim_{n->\infty} \sqrt[n]{|a_n|} = 1$, then the Root Test is inconclusive.